BINOMIAL THEOREM_SYNOPSIS

•	**Greatest term (numerically) in the expansion of $(1+x)^n$**
•	**Method 1** **Let T_r (The rth term) be the greatest term.** **Find T_{r-1}, T_r, T_{r+1} from the given expansion.** **Put** $\frac{T_r}{T_{r+1}} \geq 1$ and $\frac{T_r}{T_{r-1}} \geq 1$. **This will give an inequality from where value or values of r can be Obtained.Then, find the r th term T_r which is the greatest term.**
•	**Method 2** **Find the value of** $k = \frac{(n+1)\|x\|}{1+\|x\|}$ **If k is an integer, then T_k and T_{k+1} are equal and both are greatest terms.** **If k is not an integer, then $T_{[k]+1}$ is the greatest term, where [k] is the greatest integral part of k.**
•	**Problem solving:** **To find the greatest term in the expansion of $(x+y)^n$, write** $(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n$ **and then** **find the greatest term in** $\left(1+\frac{y}{x}\right)^n$
•	**Middle term in the Binomial expansion** **The middle term in the binomial expansion of $(x+y)^n$ depends upon the value of n.** **If n is even, then there is only one middle term, i.e.,** $\left(\frac{n}{2}+1\right)th$ **term.** **If n is odd, then there are two middle terms, i.e.,** $\left(\frac{n+1}{2}\right)th$ and $\left(\frac{n+3}{2}\right)th$ **terms.**

•	**Important points:** **When there are two middle terms in the expansion, their binomial coefficients are equal.** **Binomial coefficient of the middle term is the greatest binomial coefficient.** ***pth*** **term from the End in the Binomial Expansion of** $(x+y)^n$ ***pth*** **term from the end in the expansion of** $(x+y)^n$ **is** $(n-p+2)$ **th term from the beginning.**
•	**Properties of Binomial coefficients** **In the binomial expansion of** $(x+y)^n$**, the coefficients** $^nC_0, {}^nC_1, {}^nC_2 .... {}^nC_n$ **are denoted by** $C_0, C_1, C_2, ...C_n$ **respectively.** **If** n **is even, then greatest coefficient =** $^nC_{n/2}$ **If** n **is odd, then greatest coefficient is** $^nC_{(n-1)/2}$ or $^nC_{(n+1)/2}$

- **Explanation**

By the binomial theorem, we have

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

Or, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$

Putting x = 1, we get $2^n = C_0 + C_1 + C_2 + ... + C_n$

Therefore Sum of the binomial coefficients = 2^n

4. $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

Explanation

We have, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$

Putting x = -1, we get

$0 = C_0 - C_1 + C_2 - C_3 + C_4 - .... + (-1)^n C_n$

$\Rightarrow C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$

= $\frac{1}{2}$ **of the sum of all the coefficients =** $\frac{1}{2}.2^n = 2^{n-1}$

5. $C_0 - C_1 + C_2 - C_3 + C_4 - C_5 + ... + (-1)^n C_n = 0$

6. $C_0^{\ 2} + C_1^{\ 2} + C_2^{\ 2} + ... + C_n^{\ 2} = \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$

Explanation

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ **........ (1)**

Also, $(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + ... + C_n$ **..... (2)**

Multiplying (1) and (2), we have

$(1+x)^{2n} = [C_0 + C_1 x + C_2 x^2 + ... + C_n x^n] \times [C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + ... + C_n]$ **..... (3)**

Equating coefficients of x^n **on both sides of (3), we get**

$${}^{2n}C_n = C_0^{\ 2} + C_1^{\ 2} + C_2^{\ 2} + ... + C_n^{\ 2}$$

Hence sum of the squares of coefficients = ${}^{2n}C_n = \frac{(2n)!}{(n!)^2}$

7. $C_0^{\ 2} - C_1^{\ 2} + C_2^{\ 2} + C_3^{\ 2} .... = \begin{cases} 0, \text{if } n \text{ is odd} \\ (-1)^{n/2}.{}^nC_{n/2}, \text{ if } n \text{ is even} \end{cases}$

8. $C_0 C_1 + C_1 C_2 + C_2 C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$

9. $C_0 C_r + C_1 C_{r+1} + ... + C_{n-r} C_n = {}^{2n}C_{n-r} \ or \ {}^{2n}C_{n+r}$

10. $C_1 + 2C_2 + 3C_3 + ... + nC_n = n.2^{n-1}$

11. $C_1 - 2C_2 + 3C_3 - ...... = 0$

12. $C_n + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$

•	**Properties of** ^nC_r **If** $0 < r < n, n, r \in N$, then **1.** $r.{}^nC_r = n.{}^{n-1}C_{r-1}$ **2.** $\frac{^nC_r}{r+1} = \frac{^{n+1}C_{r+1}}{n+1}$ **3.** $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ **4.** $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$ **5.** $\frac{^nC_r}{^nC_{r+1}} = \frac{r+1}{n-r}$ **6.** $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$ **7.** $^nC_x = {}^nC_y \Rightarrow x = y \text{ or } x + y = n$ **8.** $^nC_r = {}^nC_{n-r}$ **9.** ^nC_r **is greatest if** $r = \begin{cases} n/2 \text{ if } n \text{ is even} \\ \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}$ **10. The greatest term in** $(1+x)^{2n}$ **has the greatest coefficient if** $\frac{n}{n+1} < x < \frac{n+1}{n}$.
•	**Multinomial theorem for a positive integral index** **If** $x_1, x_2, ...x_k$ **are real numbers, then for all** $n \in N$, $(x_1, x_2, ...x_k)^n$ $= \sum_{r_1+r_2+..+r_k=n} \frac{n!}{r_1!r_2!...r_k!} x_1^{r_1}.x_2^{r_2}...x_k^{r_k}$ $r_1, r_2, ...r_k$ **are all non-negative integers.** **Note:** **The general term in the above expansion is** $\frac{n!}{r_1!r_2!...r_k!} x_1^{r_1}.x_1^{r_2}...x_k^{r_k}$ **The total number of terms in the above expansion is = number of non-negative** **integral solutions of the equation** $r_1 + r_2 + .. + r_k = n = {}^{n+k-1}C_n \text{ or } {}^{n+k-1}C_{k-1}$ **The coefficient of** $x_1^{r_1}.x_1^{r_2}...x_k^{r_k}$ **in the expansion of** $(a_1x_1 + a_2x_2 + ... + a_kx_k)^n = \frac{n!}{r_1!r_2!...r_k!} a_1^{r_1}.a_2^{r_2}...a_k^{r_k}$ **Greatest coefficient in the expansion of** $(x_1 + x_2 + ... + x_k)^n = \frac{n!}{(q!)^{k-r}.[(q+1)!]^r}$ **Where *q* is the quotient and *r* the remainder when *n* is divided by *k*.**

- **PROBLEM SOLVING**

The number of terms in the expansion of $(x+y+z)^n$, where n is a positive integer, is $\frac{(n+1)(n+2)}{2}$

The number of terms in the expansion of $(x+y+z+w)^n$, where n is a positive integer, is $\frac{(n+1)(n+2)(n+3)}{6}$

Coefficient of $x^{n_1}y^{n_2}z^{n_3}$ in the expansion of $(x+y+z)^n$, is

$\frac{n!}{n_1!n_2!n_2!}$**, where** $n = n_1 + n_2 + n_3$

In the expansion of $(x_1 + x_2 + ... + x_k)^n$, the sum of all the coefficients is obtained by putting all the variables x_i equal to 1 and it is equal to k^n.

Coefficient of x^m in $(1+x^r)^n$ $(m, r\ and\ n \in N)$ is zero, if m is not an integral multiple of r, e.g., coefficient of x^{1000} in the expansion of $(1+x^3)^{4000}$ is 0 as 1,000 is not an integral multiple of 3.

Binomial Theorm for any index

If n is a rational number and x is real number such that |x| < 1, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n(n-1)(n-2)...(n-r-1)}{r!}$$

…..(1)

In the above expansion, the first term must be unity. In the expansion of $(a+x)^n$, where n is either a negative integer or a fraction, we proceed as follows.

$$(a+x)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(1+\frac{x}{a}\right)^n =$$

$$a^n\left[1 + n.\frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + ...\right]$$

And the expansion is valid when $\left|\frac{x}{a}\right|$ < 1 i.e |x| < |a|.

There are infinite number of terms in the expansion of $(1+x)^n$, When n is a negative integer or a fraction.

If x is so small that its square and higher powers may be neglected, then approximate value of $(1+x)^n = 1 + nx$.

- **General term in the expansion of $(1+x)^n$**

The $(r+1)th$ term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)....(n-r-1)}{r!}x^r$$

- **Some Important Deductions from $(1+x)^n$**

	Replacing *n* by –*n* in (1), we get $(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+....+(-1)^r\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r+...$ $T_{r+1}=(-1)^r\frac{n(n+1)(n+2)....(n+r-1)}{r!}x^r$ **Replacing *x* by –*x* in (1), we get** $(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+....+(-1)^r\frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r+...$ $T_{r+1}=(-1)^r\frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r$ **Replacing x by –x and n by –n in (1), we get** $(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+....+(-1)^r\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r+...$ $T_{r+1}=(-1)^r\frac{n(n+1)(n+2)....(n+r-1)}{r!}x^r$
•	**Some useful expansions** **1.** $(1+x)^{-1}=1-x+x^2-x^3+....+(-1)^r x^r+...$ **2.** $(1-x)^{-1}=1+x+x^2+x^3+....+x^r+...$ **3.** $(1+x)^{-2}=1-2x+3x^2-4x^3+....+(-1)^r(r+1)x^r+...$ **4.** $(1-x)^{-2}=1+2x+3x^2+4x^3+....+(r+1)x^r+...$ **5.** $(1+x)^{-3}=1-3x+6x^2-10x^3+....+(-1)^r\frac{(r+1)(r+2)}{2}x^r+...$ **6.** $(1-x)^{-3}=1+3x+6x^2+10x^3+....+\frac{(r+1)(r+2)}{2}x^r+...$

BINOMIAL THEOREM_ASSIGNMENT

1. If $C_0, C_1, C_2, ...C_n$ are the coefficients of the expansion of $(1+x)^n$, then the value of $\sum_{0}^{n} \frac{C_k}{k+1}$ is

   a) 0 b) $\frac{2^n-1}{n}$ c) $\frac{2^{n+1}-1}{n+1}$ d) None of these

2. Larger of $99^{50}+100^{50}$ and 101^{50} is

   a) 101^{50} b) $99^{50}+100^{50}$ c) Both are equal d) None of these

3. The greatest coefficient in the expansion of $(x+y+z+w)^{15}$ is

   a) $\frac{15!}{3!(4!)^3}$ b) $\frac{15!}{(3!)^3 4!}$ c) $\frac{15!}{2!(4!)^2}$ d) None of these

4. The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is

   a) $2^{19}-\frac{1}{2}.{}^{20}C_{10}$ b) $2^{19}+\frac{1}{2}.{}^{20}C_{10}$ c) 2^{19} d) 2^{20}

5. ${}^{n+1}C_2 + 2[{}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + ... + {}^{n}C_2] =$

   a) $\frac{n(n+1)(2n+1)}{6}$ b) $\frac{n(n+1)}{2}$ c) $\frac{n(n-1)(2n-1)}{6}$ d) None of these

6. If $A = {}^{2n}C_0 . {}^{2n}C_1 + {}^{2n}C_1 \, {}^{2n-1}C_1 + {}^{2n}C_2 \, {}^{2n-2}C_1 + ...$ then A is

   a) 0 b) 2^n c) $n2^{2n}$ d) 1

7. The greatest integer which divides the number $101^{100}-1$ is

   a) 100 b) 1,000 c) 10,000 d) 1,00,000

8. If $\{x\}$ denotes the fractional part of x, then $\left\{\frac{2^{2003}}{17}\right\}$ is

   a) $\frac{2}{17}$ b) $\frac{4}{17}$ c) $\frac{8}{17}$ d) $\frac{16}{17}$

9. If [x] denotes the greatest integer less than or equal to x, then $[(6\sqrt{6}+14)^{2n+1}]$

   a) Is an even integer b) Is an odd integer c) Depends on n d) None of these

10. The number of distinct terms in the expansion of $\left(x^3+1+\frac{1}{x^3}\right)^n; x\in R^+$ and $n\in N$ is

a) 2n b) 3n c) 2n + 1 d) 3n + 1

11. The sum to (n + 1) terms of the series $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+\ldots\ldots\ldots$ is

a) $\frac{1}{n(n+1)}$ b) $\frac{1}{n+2}$ c) $\frac{1}{n+1}$ d) None of these

12. The integral part of $(8+3\sqrt{7})^n$ is

a) An even integer b) An odd integer c) Zero d) Nothing can be said

13. Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$ where [] denotes the greatest integer function. Then $Rf=$

a) 2^{2n+1} b) 2^{4n+1} c) 4^{2n+1} d) None of these

14. If p is nearly equal to q and $n > 1$, then $\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=$

a) $\left(\frac{p}{q}\right)^n$ b) $\left(\frac{q}{p}\right)^n$ c) $\left(\frac{p}{q}\right)^{1/n}$ d) None of these

15. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q}$ = 0 if $p > q$) is maximum when m is….

a) 20 b) 15 c) 12 d) 10

16. The number of integral terms in the expansion of $(2\sqrt{5}+\sqrt[6]{7})^{642}$ is….

a)105 b) 104 c) 108 d) 107

17. If the fourth term in the expansion of $\left(\sqrt{\frac{1}{x^{\log x+1}}}+x^{1/12}\right)^6$ is equal to 200 and $x > 1$, then x is equal to ….

a) 10 b) 15 c) 5 d) 20

18. The number of rational terms in the expansion of $(1+\sqrt{2}+\sqrt[3]{5})^6$ is

a) 13 b) 15 c) 19 d) 7

19. Let $n \in N$ and $n < (\sqrt{2}+1)^6$. Then, the greatest value of n is …..

a) 197 b) 193 c)185 d) 183

20. The coefficient of x^5 in the expansion of $(1+2x+3x^2)^4$ is

a) 300 b)308 c) 312 d) 316

21. The coefficient of x^{-1} in the expansion of $(1+3x^2+x^4)\left(1+\frac{1}{x}\right)^8$ is……

a) 232 b) 226 c) 220 d) 217

22. If the last term in the binomial expansion of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is…….

a) 0.21 b) 21 c) 210 d) 2100

23. The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$ is

a) 975 b) 985 c) 990 d) 995

24. The coefficient of the term independent of x in the expansion of

$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is

a) 210 b) 110 c) 310 d) 410

25. The coefficient of $x^2y^3z^5$ in the expansion of $(x+y+z)^{10}$ is

a) 2520 b) 25.2 c) 2.520 d) 0.2520

26. If $(1+x+x^2)^n = a_0+a_1x+a_2x^2+.....+a_{2n}x^{2n}$, then $a_0+a_3+a_6+....=$

a) 3^{n+1} b) 3^n c) 3^{n-1} d) None of these

27. $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+\frac{C_6}{7}+.....=$

a) $\frac{2^n}{n}$ b) $\frac{2^{n-1}}{n}$ c) $\frac{2^n}{n+1}$ d) None of these

28. If $(1+x)^{15} = C_0+C_1x+C_2x^2+....+C_{15}x^{15}$, then the value of $C_2+2C_3+3C_4+....+14C_{15}$ is

a) 219923 b) 16789 c) 219982 d) None of these

29. If $a_0, a_1, a_2, \ldots\ldots, a_{2n}$ be the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, then $a_0^{\,2} - a_1^{\,2} + a_2^{\,2} - a_3^{\,2} + \ldots\ldots - a_{2n-1}^{\,2} + a_{2n}^{\,2} =$

a) a_{2n} b) a_n c) a_0 d) None of these

30. The coefficient of x^{50} in the expression $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots\ldots + 1001x^{1000}$ is

a) ${}^{1000}C_{50}$ b) ${}^{1001}C_{50}$ c) ${}^{1002}C_{50}$ d) None of these

31. The sum of the series $\sum_{r=0}^{n}(-1)^r.{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots\ldots to\ m\ terms\right]$ is

a) $\frac{1-\frac{1}{2^{mn}}}{2^m-1}$ b) $\frac{1-\frac{1}{2^{mn}}}{2^n-1}$ c) $\frac{1-\frac{1}{2^m}}{2^n-1}$ d) None of these

32. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots. + C_nx^n$, then for n even, $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - \ldots. + (-1)^n C_n^{\,2}$ is equal to

a) 0 b) $(-1)^{n/2}\,{}^nC_{n/2}$ c) ${}^nC_{n/2}$ d) None of these

KEY SHEET

1)c	2)a	3)a	4)b	5)a	6)c	7)c	8)c	9)a	10)c
11)d	12)b	13)c	14)c	15)b	16)c	17)a	18)d	19)a	20)c
21)a	22)c	23)c	24)a	25)a	26)c	27)c	28)a	29)b	30)b
31) B	32) B								